





HD28  
.M414

no. 3437  
-92

**HOLDING COSTS AND EQUILIBRIUM ARBITRAGE**

**BY**

**BRUCE TUCKMAN AND JEAN-LUC VILA**  
**LATEST VERSION: JUNE 1992**  
**WORKING PAPER NO: 3437-92-EFA**



**HOLDING COSTS AND EQUILIBRIUM ARBITRAGE**

**BY**

**BRUCE TUCKMAN AND JEAN-LUC VILA**

**LATEST VERSION: JUNE 1992**

**WORKING PAPER NO: 3437-92-EFA**

M.I.T. LIBRARIES  
AUG 27 1992  
REC'D 7 1992  
RECEIVED



# **HOLDING COSTS AND EQUILIBRIUM ARBITRAGE**

**Bruce TUCKMAN**

*Stern School of Business, New York University, New York, NY 10006, USA*

**Jean-Luc VILA**

*Sloan School of Management, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

**June, 1992**



## Abstract

In a world where trading is costless, assets with identical cash flows must have identical prices. If arbitrageurs face unit time costs, or holding costs, the prices of these assets need not be equal, i.e. the assets can be relatively mispriced. This paper constructs a dynamic model of the equilibrium determination of prices under costly arbitrage. Our analysis reveals that:

- (i) Mispricing and arbitrage can exist in a market equilibrium.
- (ii) Riskless arbitrage arguments may not provide tight bounds around observed market prices.
- (iii) Arbitrage activity reduces equilibrium mispricing and is particularly effective when liquidity shocks are transient and conditionally volatile.

Acknowledgments: We would like to thank John Heaton for helpful comments and suggestions. Also Jean-Luc Vila wishes to acknowledge financial support from the International Financial Services Research Center at the Sloan School of Management.



## 1. Introduction

The notion of arbitrage is extremely useful in markets without frictions. Researchers define arbitrage as a set of transactions which costs nothing and yet risklessly provides positive cash flows. They then assume that arbitrage opportunities never exist and derive precise implications about relative prices.

The presence of market frictions, however, reduces the usefulness of arbitrage arguments. In the simplest of models, trading costs lower arbitrage profits. Therefore, when market prices do not admit riskless profit opportunities inclusive of trading costs, arbitrageurs do nothing. When market prices do admit such opportunities, arbitrageurs stake all they have on the sure proposition that prices will eventually come back into line.

Richer models of arbitrage activity recognize that arbitrageurs may take positions even if the simple, buy-and-hold arbitrage strategy does not furnish riskless profit opportunities. In one such model,<sup>1</sup> Tuckman and Vila (1992a) constrain market mispricings so that they never violate the riskless arbitrage bounds. Nevertheless, risk averse arbitrageurs facing unit-time costs, or holding costs, take a finite, risky position if and only if mispricings are large enough. This result raises the possibility that arbitrageurs routinely find and take advantage of market mispricings, i.e. arbitrage may exist in a market equilibrium.

This paper embeds the Tuckman and Vila (1992a) model in a dynamic, equilibrium setting in order to explore the effects of market frictions and arbitrage activity on equilibrium prices. The model assumes that there are two distinct bond issues which generate identical cash flows over time. Each bond issue trades in its own market. Furthermore, while the two markets are identical in all respects, they are segmented in that investors in one market cannot trade in the other. While under these assumptions the two bonds would sell for the same price, adding random liquidity shocks to

---

<sup>1</sup> Brennan and Schwartz (1990) and Hodges and Neuberger (1989) are other examples.

each market drives the prices of the bonds apart. Finally, risk-averse arbitrageurs facing holding costs trade across the two markets to take advantage of any price deviations which do occur.

Many lessons emerge from the equilibrium prices determined through this model. First, because arbitrageurs do not take large enough positions to bring prices immediately back into line, mispricings and arbitrage activity are consistent with market equilibrium. Second, because arbitrageurs take positions even when no riskless profit opportunities are available, equilibrium prices are often well within the riskless arbitrage bounds. This finding implies that models based on riskless arbitrage arguments, like many in the option replication literature,<sup>2</sup> may not provide usefully tight bounds on observed prices. Third, arbitrage activity reduces both the magnitude and unconditional variance of the mispricing process.<sup>3</sup> Fourth, arbitrageurs are most effective in eliminating mispricings when the liquidity shocks are transient and conditionally volatile. This last finding serves as a microeconomic foundation for the recent literature on the connection between investor impatience and market mispricings. De Long *et al.* (1990) and Lee *et al.* (1991) have hypothesized that investors' short term horizons allow for persistent deviations from fundamental values. Shleifer and Vishny (1990) argue that when mispricings increase in asset maturity, risk averse corporate managers will pursue short term objectives. The present analysis reveals that the nature of holding costs cause arbitrageurs to react most forcefully to mispricings stemming from causes which tend to disappear relatively quickly.

This paper contributes not only to the arbitrage pricing literature, but also to the growing literature on imperfect financial markets. While the impact of frictions such as trading costs,<sup>4</sup> holding costs,<sup>5</sup> borrowing constraints,<sup>6</sup> or market incompleteness<sup>7</sup> on dynamic investment strategy is relatively

---

<sup>2</sup>See, for example, Leland (1985) and Boyle and Vorst (1992).

<sup>3</sup> See Grossman and Miller (1988), Fremault (1991) and Holden (1990) for similar results in a static framework.

<sup>4</sup>See Constantinides (1986), Davis and Norman (1990), Davis, Panas and Zariphopoulou (1991), Duffie and Sun (1989), Fleming *et al.* (1989).

<sup>5</sup>See Tuckman and Vila (1992).

well understood, little is known about the impact of these frictions on equilibrium price processes. While some recent papers have addressed this question, they focus on trading costs rather than holding costs.<sup>8</sup>

Finally, this paper has important welfare implications. Because most investors face relatively high transaction costs, their activities alone cannot ensure that marginal rates of return are equated across markets. While arbitrageurs are better suited for this function, they face some frictions as well. Therefore, this paper's analysis can be viewed as a study which relates a common market imperfection to the inefficiencies it generates.

Section I describes holding costs and how they affect arbitrage activity. It also argues that holding costs are important in many different arbitrage contexts.

Section II describes the model. The first subsection discusses the essential elements of any model of equilibrium arbitrage activity. The second subsection develops the equilibrium model without arbitrageurs. The third subsection develops the equilibrium model in the presence of arbitrageurs.

Section III demonstrates the solution method for the model. Section IV presents a numerical example which explores the model's implications. Section V concludes and suggests avenues for future research.

---

<sup>6</sup>See Grossman and Vila (1992), Vila and Zariphopoulou (1990), He and Pearson (1991 a and b).

<sup>7</sup>See He and Pearson (1991 a and b).

<sup>8</sup>See Ayagari and Gertler (1991), Heaton and Lucas (1992) and Vayanos and Vila (1992).

## 2. Holding Costs and Risky Arbitrage

Markets provide many examples of distinct portfolios which generate identical cash flows. Some common instances are a forward or futures contract vs. a levered position in the spot asset, a bond denominated in one currency vs. a bond denominated in another currency plus a cross-currency forward contract, and a coupon bond vs. a cash-flow matched portfolio of other coupon bonds.

Consider two such portfolios, one "red" and one "green." Although they generate identical cash flows, assume that, for some reason, the market prices of the portfolios,  $P_R$  and  $P_G$ , differ by an amount  $\Delta$ . Assuming that  $P_R > P_G$ , arbitrageurs facing no frictions will short the red portfolio and purchase the green portfolio, realizing a riskless profit of  $\Delta$ .

In reality, short-sale agreements are more complex. In order to short the red portfolio, an arbitrageur must borrow the securities from someone willing to lend them. Furthermore, the lender will demand collateral in order to protect his position. Therefore, the arbitrage transaction will be arranged in the following manner: lend  $P_R$  to the holder of the securities in the red portfolio (or, equivalently, post an interest-bearing security worth  $P_R$ ), take the securities as collateral and sell them for  $P_R$ , purchase the green portfolio, and borrow  $P_G$ . This set of transactions generates no cash flows today, but assures the realization of the future value of  $\Delta$  upon closing the position.

If markets were frictionless, the equilibrium value of  $\Delta$  must equal 0. Were it not so, arbitrageurs would take infinite quantities of the arbitrage position described above. Holding costs change this story in two important ways. First, ruling out riskless arbitrage requires only that  $\Delta$  be less than or equal to the present value of the maximum possible holding costs. Second, arbitrageurs face a *risky* investment opportunity when confronted with smaller values of  $\Delta$ : if  $\Delta$  vanishes quickly enough, the costs of the arbitrage will be small and the transaction will have proved profitable. If, on the other hand,  $\Delta$  does not vanish quickly enough, the total holding costs will swamp the realized gains.

Holding costs appear in many arbitrage contexts. First, shorting any spot security or



commodity will often result in unit time costs since arbitrageurs must usually sacrifice the use of at least some of the short sale proceeds. It is true that, in the case of stocks, brokerage houses with large client bases assume short positions almost costlessly. Nevertheless, smaller arbitrageurs do incur holding costs when short-selling stocks. Second, futures market positions may generate unit time costs since part of margin deposits may not earn interest. Third, banks making markets in forward contracts often charge a *per annum* rate over the life of the contract. Fourth, when collateral requirements cause deviations from desired investment strategies, these requirements are essentially generating unit time costs.<sup>9</sup>

### 3. The Model

#### 3.1. Preliminaries

Any model of equilibrium arbitrage must 1) posit the existence of two distinct assets which generate identical cash flows, 2) assume some forces which cause the prices of these assets to differ, and 3) restrict investor and arbitrageur behavior so that price differences do not vanish as soon as they appear.

As discussed in the previous section, there are many examples of distinct assets which generate identical cash flows. This model assumes that there exist two distinct bond issues with identical coupons and maturities. For ease of exposition, one of the issues will consist of "red" bonds, while the other will consist of "green" bonds. Not much effort would be required to recast the model in terms of the other mentioned examples.

The prices of the red and green bonds may tend to differ for a number of reasons. For

---

<sup>9</sup> See Tuckman and Vila (1992) for references supporting these institutional details. Also see that paper for a detailed description of the magnitude of holding costs in the U.S. Treasury bond market.

example, the bonds might be traded by different clienteles with different valuation rules. Clienteles of this sort may have developed for historical reasons or may exist for institutional reasons. Another reason for price differences across these markets might be temporary supply and demand imbalances due to microstructure imperfections. In any case, because this paper aims at explaining the process which constrains these price differences, no serious effort has been made to model the source of price differences. Instead, the model assumes that noise traders occasionally shock the red and green bond markets with buy and sell orders. To ensure that these shocks affect market prices, it is furthermore assumed that the demand for the individual bond issues is not perfectly elastic.

Because close substitutes exist for most financial assets, the more usual assumption is that the demand for individual financial assets is perfectly elastic. But, in the presence of market frictions, considerations other than the existence of substitutes become important. For example, investors in different tax brackets value the same asset differently, thus generating a downward-sloping demand curve. Similarly, to the extent that an asset is purchased after the sale of other assets, investors with low transaction costs will buy at higher prices than investors with high transaction costs, again leading to a downward-sloping demand curve. This model employs the tax motivation because, as shown below, the functional form of the demand curve can be easily derived. But, note that any downward-sloping demand curve will translate liquidity shocks into price differences across markets.

Finally, price differences must not vanish as they appear. This certainly requires some segmentation of the markets; if investors in one bond market can easily purchase bonds in the other, price differences will result in migrations from the relatively dear market to the relatively cheap market. And these migrations will, in turn, equalize the red and green bond prices. While market segmentation seems a reasonable assumption from the point of view of many investors, arbitrageurs can usually trade across markets. Nevertheless, arbitrageur activity might not be sufficient to force prices back into line. Possible assumptions which limit arbitrageur positions are exogenous position

limits (see Brennan and Schwartz (1990)), oligopolistic behavior (see Holden (1990)), or non-synchronous trading across the two markets (see Fremault (1990) and (1991)). Here, arbitrageurs are assumed to be risk-averse and to face holding costs when shorting bonds. As in Tuckman and Vila (1992a), these assumptions imply that optimal arbitrage positions are not necessarily large enough to eliminate price differences across markets.

### 3.2. *Equilibrium pricing without arbitrageurs*

Turning to the model of this paper, begin by assuming that there are two distinct bond issues, one red and one green, trading in two distinct markets. Both bond issues mature  $n$  periods from now and entitle holders to  $\$d$  at the end of each period before maturity and to  $\$1+d$  at maturity. The coupon payments are taxable to investors at their own tax rates, but there are no capital gains taxes. Finally, investors who buy bonds plan to hold them until maturity.<sup>10</sup> Then, letting  $r$  denote the discount rate for after-tax cash flows, the value of a bond to an investor with tax rate  $\tau$ , in either market, is

$$\begin{aligned} V_n(\tau) &= \frac{d(1-\tau)}{r} \left[ 1 - \frac{1}{(1+r)^n} \right] + \frac{1}{(1+r)^n} \\ &= V_n(0) - \frac{\tau d}{r} \left[ 1 - \frac{1}{(1+r)^n} \right]. \end{aligned} \tag{1}$$

For a given bond price,  $P_n$ , an investor with tax rate  $\tau$  will want to buy bonds if  $P_n < V_n(\tau)$ .

Solving (1) for  $\tau$  shows that investors with tax rate  $\tau$  will want to buy bonds so long as

---

<sup>10</sup> This assumes that investors are myopic in the following sense. When deciding to buy or sell a bond, they compare the market price to their own valuation under a buy and hold strategy. But this strategy is not necessarily optimal, since investors may prefer to delay a sale in the expectation that prices will rise. While one might be tempted to think of the decision to sell in terms of exercising an option, the analogy is misleading: an investor who sells, i.e. exercises, can repurchase the bond later and sell yet again. In fact, in a related context, Tuckman and Vila (1992b) show that the myopic strategy is sometimes optimal. In any case, careful modeling of investor decisions in the context of the present model will be left as a subject for future research.

$$\tau < \frac{r}{d} \frac{V_n(0) - P_n}{1 - \frac{1}{(1+r)^n}}. \quad (2)$$

To derive a demand curve for the bonds, assume that i) investors buy at most 1 bond and ii) the number of investors with tax rates below some  $\tau$  in each of the red and green bond markets is given by  $\alpha\tau$ . From these assumptions and (2), the demand function in each market,  $D_n(P_n)$ , is given by

$$D_n(P_n) = \frac{\alpha r}{d} \frac{V_n(0) - P_n}{1 - \frac{1}{(1+r)^n}}. \quad (3)$$

Assume that the quantity of each bond issue available for investor trading is  $Q$ . In the absence of noise traders, the market price in each bond market would be the  $P_n$  which solves  $D_n(P_n) = Q$ . The presence of noise traders, however, can cause the prices of the red and green bonds to differ.

Noise traders enter the markets to sell or to buy. Let  $L_{R,n}$  and  $L_{G,n}$  be the *cumulative* amount of noise trading in the red and green bond markets, respectively. By convention, positive quantities denote supply shocks while negative quantities denote demand shocks. If, for instance,  $L_{R,n} > L_{R,n+1}$  there are  $L_{R,n} - L_{R,n+1}$  sellers in the market for red bonds. If, on the other hand,  $L_{R,n} < L_{R,n+1}$ , there are  $L_{R,n+1} - L_{R,n}$  buyers. As in other papers,<sup>11</sup> no explicit model of the sources of noise trading will be presented. Finally, note that noise trading shocks affect the quantity of bonds available for investor trading: the supply of red bonds after the shock is  $Q + L_{R,n}$  while the supply of green bonds after the shock is  $Q + L_{G,n}$ .

In the absence of arbitrageurs, the equilibrium prices in the red and green markets,  $P_{R,n}$  and  $P_{G,n}$ , respectively, are determined by the following equations:

---

<sup>11</sup>See for instance, Kyle (1985), DeLong *et. al.* (1990), Fremault (1990 and 1991) and Holden (1990).

$$\frac{\alpha r}{d} \frac{V_n(0) - P_{R,n}}{1 - \frac{1}{(1+r)^n}} = Q + L_{R,n} \quad \text{and} \quad (4a)$$

$$\frac{\alpha r}{d} \frac{V_n(0) - P_{G,n}}{1 - \frac{1}{(1+r)^n}} = Q + L_{G,n} . \quad (4b)$$

Solving for the prices,

$$P_{R,n} = V_n(0) - \theta_n (Q + L_{R,n}) \quad (5a)$$

$$P_{G,n} = V_n(0) - \theta_n (Q + L_{G,n}) \quad (5b)$$

where

$$\theta_n = \frac{d}{\alpha r} \left[ 1 - \frac{1}{(1+r)^n} \right] . \quad (6)$$

Of particular interest is the difference between the price of red bonds and the price of green bonds. Letting  $\Delta_n = P_{R,n} - P_{G,n}$  and using the pricing equations (5), the relative mispricing equals

$$\Delta_n = \theta_n L_n \quad (7)$$

with

$$L_n = L_{G,n} - L_{R,n}.$$

### 3.3. Equilibrium pricing with arbitrageurs

Arbitrageurs can now be introduced into the model. Let  $x_n$  be the number of green bonds bought by arbitrageurs and the number of red bonds sold by arbitrageurs. The optimal choice of  $x_n$  will be discussed below. For now, consider the effect of arbitrage on the mispricing  $\Delta_n$ . If  $L_n > 0$  and  $\Delta_n > 0$ , i.e. if  $P_{R,n} > P_{G,n}$ , arbitrageurs will want to sell red bonds and buy green bonds, so  $x_n$  will be positive. Furthermore,  $x_n$  will be added to the supply of red bonds and added to the demand of green bonds. Adjusting equations (5a) and (5b) accordingly and subtracting (5b) from (5a) to obtain  $\Delta_n$  for this case,  $\Delta_n = \theta_n (L_n - 2x_n)$ . Notice that arbitrage activity lowers the relative mispricing.

If  $L_n < 0$  and  $\Delta_n < 0$ , i.e.  $P_{R,n} < P_{G,n}$ , arbitrageurs will want to buy red bonds and sell green bonds, so  $x_n$  will be negative. Furthermore,  $x_n$  will be added to the demand for red bonds and to the supply of green bonds. In this case also  $\Delta_n = \theta_n (L_n - 2x_n)$  and arbitrage activity reduces the relative mispricing.

Summarizing this discussion, adjusting supply and demand in the bond markets to account for arbitrageur activity changes the mispricing  $\Delta_n$  from (7) to

$$\Delta_n = \theta_n (L_n - 2x_n), \quad (8)$$

By definition,  $x_n$  is the sum of positions across arbitrageurs. Therefore, an equilibrium in this model consists of a strategy  $\{x_n\}$  and a process  $\{\Delta_n\}$  such that 1) each arbitrageur chooses an optimal strategy given the evolution of  $\{\Delta_n\}$ , and 2) the resulting  $\{x_n\}$ , in turn, generates the process  $\{\Delta_n\}$  given by (8).

For simplicity, assume that all arbitrageurs have negative exponential utility functions and that they maximize their expected utilities of wealth as of the date the bonds mature. In that case, there exists a representative arbitrageur with negative exponential utility who maximizes expected utility of terminal wealth. To solve for equilibrium prices, then, one must solve the investment problem of the representative arbitrageur.

Begin with the value of an arbitrage position from one period to the next. Assume for the moment that  $\Delta_n > 0$ , i.e.  $P_{R,n} > P_{G,n}$ . As discussed in the previous section, the arbitrageur will buy  $x_n$  green bonds, borrow  $x_n P_{G,n}$  dollars, short  $x_n$  red bonds, and lend  $x_n P_{R,n}$  dollars,  $x_n > 0$ . Next period the position is worth

$$x_n P_{G,n+1} - x_n P_{R,n+1} + x_n P_{R,n}(1+r) - x_n P_{G,n}(1+r) - c|x_n| = x_n [\Delta_n(1+r) - \Delta_{n+1}] - c|x_n|, \quad (9)$$

where  $c$  is the holding cost incurred for maintaining a short position over the period. If  $\Delta_n < 0$ , the expression does not change, but  $x_n$  will be negative: the arbitrage position entails buying green bonds and shorting red bonds.

Let  $W_n$  be the wealth of the representative arbitrageur following the strategy  $\{x_n\}$  when there are  $n$  periods to maturity. From the above discussion, his wealth one period later,  $W_{n-1}$ , is

$$W_{n-1} = W_n(1+r) + x_n [\Delta_n(1+r) - \Delta_{n-1}] - c|x_n|. \quad (10)$$

Defining  $w_n = W_n(1+r)^n$ ,  $\delta_n = \Delta_n(1+r)^n$ , and  $c_n = c(1+r)^n$ , (10) can be rewritten as

$$w_{n-1} = w_n + x_n [\delta_n - \delta_{n-1}] - c_{n-1}|x_n|. \quad (11)$$

Equation (11) and the objective, discussed above, to maximize the expected value of  $-\exp(-Aw_0)$ ,  $A > 0$ , completes the specification of the representative arbitrageur's investment problem given  $\{\Delta_n\}$ .

The model can be completed by specifying an exogenous stochastic process for the net liquidity shocks,  $\{L_n\}$ . For simplicity it will be assumed that  $\{L_n\}$  evolves as a binomial process: if its value with  $n$  periods to maturity is  $L_n$ , then it will take on the value  $L_{n-1} = L_n + u$  with probability  $\pi(n, L_n)$  and a value  $L_{n-1} = L_n - u$  with probability  $1 - \pi(n, L_n)$ .

### 3.4. Model Solution

This section begins by solving the representative arbitrageur's investment problem. Let  $V(w_n, L_n, n) = \max E_n[-\exp(-Aw_0)]$  where  $E_n$  denotes the expectation when there are  $n$  periods to maturity and the maximum is over the strategy  $\{x_n\}$ . By the principle of optimality in dynamic programming,

$$V(w_n, L_n, n) = \max_{x_n} \{ \pi(n, L_n) V(w_n + x_n [\delta_n(L_n) - \delta_{n-1}(L_n + u)] - c_{n-1}|x_n|, L_n + u, n-1) + (1 - \pi(n, L_n)) V(w_n + x_n [\delta_n(L_n) - \delta_{n-1}(L_n - u)] - c_{n-1}|x_n|, L_n - u, n-1) \} \quad (12)$$

Also, because the mispricing must vanish at the maturity date, the initial condition of the problem is  $V(w_0, L_0, 0) = -\exp(-Aw_0)$ .

Because of the special form of the utility function,  $V(w_n, L_n, n)$  is separable in wealth and the

arbitrage opportunity and can be written as  $\exp\{-Aw_n\}J(L_n, n)$ . Using this fact, (12) becomes

$$J(L_n, n) = \max_{x_n} \{ \pi(n, L_n) e^{-A[x_n(\delta_n(L_n) - \delta_{n-1}(L_n+u)) - c_{n-1}|x_n|]} J(L_n+u, n-1) + (1 - \pi(n, L_n)) e^{-A[x_n(\delta_n(L_n) - \delta_{n-1}(L_n-u)) - c_{n-1}|x_n|]} J(L_n-u, n-1) \} \quad (13)$$

with initial condition  $J(L_0, 0) = -1$ .

To solve for the optimal strategy as a function of  $\delta_n$ , begin as follows. If  $\delta_n > 0$ , the mispricing increases with a move to  $\delta_{n-1}(L_n+u)$  while the mispricing decreases with a move to  $\delta_{n-1}(L_n-u)$ . Since the per period arbitrage profits is  $x_n(\delta_n - \delta_{n-1} - c_{n-1})$ , the position will never be profitable if  $\delta_n < \delta_{n-1}(L_n-u) + c_{n-1}$ , i.e. if the position is not profitable even when the relative mispricing falls. So, for these value of  $\delta_n$ ,  $x_n = 0$ . On the other hand,  $\delta_n < \delta_{n-1}(L_n+u) + c_{n-1}$  is inconsistent with equilibrium; if the position is profitable even when the relative mispricing rises, prices furnish a riskless arbitrage inclusive of holding costs and the optimal  $x_n$  would equal  $+\infty$ . In the intermediate range,

$$\delta_{n-1}(L_n+u) + c_{n-1} > \delta_n > \delta_{n-1}(L_n-u) + c_{n-1}$$

$x_n$  can be found by solving the optimization problem (13) to obtain

$$x_n = \left[ \frac{1}{A[\delta_{n-1}(L_n+u) - \delta_{n-1}(L_n-u)]} \text{Ln} \left\{ \frac{1 - \pi(n, L_n)}{\pi(n, L_n)} \frac{\delta_n(L_n) - \delta_{n-1}(L_n-u) - c_{n-1}}{c_{n-1} + \delta_{n-1}(L_n+u) - \delta_n(L_n)} \frac{J(L_n-u, n-1)}{J(L_n+u, n-1)} \right\} \right]^+ \quad (14)$$

where  $(H)^+ = \text{Max} \{H, 0\}$ .

If  $\delta_n < 0$ , similar arguments reveal that  $x_n = 0$  when  $\delta_n > \delta_{n-1}(L_n+u) - c_{n-1}$  while  $x_n = -\infty$  when  $\delta_n < \delta_{n-1}(L_n-u) - c_{n-1}$ . In the intermediate range,

$$\delta_{n-1}(L_n+u) - c_{n-1} > \delta_n > \delta_{n-1}(L_n-u) - c_{n-1}$$

the optimal  $x_n$  is given by

$$x_n = \left[ \frac{1}{A[\delta_{n-1}(L_n+u) - \delta_{n-1}(L_n-u)]} \text{Ln} \left\{ \frac{1 - \pi(n, L_n)}{\pi(n, L_n)} \frac{\delta_n(L_n) - \delta_{n-1}(L_n-u) + c_{n-1}}{-c_{n-1} + \delta_{n-1}(L_n+u) - \delta_n(L_n)} \frac{J(L_n-u, n-1)}{J(L_n+u, n-1)} \right\} \right]^- \quad (15)$$

where  $(H)^- = \text{Min} \{H, 0\}$

To solve for the equilibrium values of  $x_n$  and  $\delta_n$ , rewrite the equilibrium condition (8) in terms



of  $\delta_n$ :

$$\delta_n = \theta_n (1+r)^n (L_n - 2x_n) . \quad (16)$$

Then, solve (16) simultaneously with (14) or (15), as appropriate. For the case  $\delta_n > 0$ , figure 1 illustrates the optimal position size and the equilibrium condition as a function of the mispricing  $\delta_n$ . The dotted line represents the mispricing which generates infinite arbitrage activity. The simultaneous solution is given by the intersection of the two functions.

#### INSERT FIGURE 1

Given the solution technique for any  $n$  given the values at  $n-1$ , backward induction will provide the solution for all  $n$ . The initial condition of the problem gives  $J(\cdot, 0)$ . This allows for the solution of  $x_1$  and  $\delta_1$  along the lines described above. Then, substituting these values into (13) yields  $J(\cdot, 1)$ . Proceeding in this fashion produces the entire mispricing process and the accompanying arbitrage strategy.

#### 4. A Numerical Example

In order to illustrate the insights of the model, this section presents the solution of the model with the following parameter values:

$d = 8\%$  (annual bond coupon rate)

$r = 8\%$  (annual after-tax discount factor)

$\alpha = 100$  (number of investors in each market)

$c = .5\%$  (annual holding cost)

$A = .0001$  (representative arbitrageur's coefficient of absolute risk aversion)

$T = 5$  years (maturity of the bond as of the starting date)

$L_T = 0$  (initial net liquidity shock)

Next, adjust the parameters to allow for different trading increments:

$h =$  length of the trading period

$N = \frac{T}{h} =$  number of periods to maturity as of starting date

$r_h = (1+r)^h - 1 =$  per period interest rate

$d_h = d r_h / r =$  per period coupon rate<sup>12</sup>

$c_h = c r_h / r =$  per period holding cost.

The discretization above preserves the present value of the coupon and of the holding cost in the sense that

$$\sum_{k=1}^{1/h} \frac{d_h}{(1+r_h)^k} = \frac{d}{1+r} \quad \text{and}$$

$$\sum_{k=1}^{1/h} \frac{c_h}{(1+r_h)^k} = \frac{c}{1+r} .$$

Finally, define the stochastic evolution of the liquidity shock,  $L_n$ , by setting

$$u_b = \sigma \sqrt{h} \quad \text{and}$$

$$\pi(n, L_n) = \max \{ \min \{ \frac{1}{2} (1 - \frac{\rho L}{\sigma} \sqrt{h}) ; -1 \} ; 1 \} ; \quad \rho > 0 .$$

The process  $L_n$  is a discretized version of the Ornstein-Uhlenbeck process

$$dL_t = -\rho L_t dt + \sigma db_t$$

where  $db_t$  is a Brownian Motion. In particular, it can be shown that the conditional expectation of the incremental liquidity shock  $(L_{n+1} - L_n)$  equals  $-\rho L_n h + o(h)$  while its conditional variance equals  $\sigma^2 h + o(h)$ .<sup>13</sup> For the analysis that follows it is useful to recall that, if  $\rho$  is positive, the Ornstein-Uhlenbeck process has a stationary distribution which is normal with mean 0 and variance  $\sigma^2 / 2\rho$ . Hence the unconditional volatility of the liquidity shock comes from two sources, the conditional standard deviation  $\sigma$  and the persistence measure  $1/\rho$ . As shown below, arbitrage activity affects both sources of unconditional volatility.

---

<sup>12</sup>This assume that coupons are paid each period.

<sup>13</sup>For convergence results on this discretization scheme see Nelson and Ramaswamy (1990).

The rest of this section is devoted to the effects of arbitrage activity on the relative mispricing of the bonds. The first subsection presents basic properties of the optimal arbitrage strategy  $x_n$  and its effects on the mispricing process. The second subsection analyzes the effects of arbitrage on the dynamics of the mispricing process, i.e. on the conditional moments of the mispricing.

#### 4.1. Basic results

For the purpose of this subsection, it is sufficient to consider the case where the cumulative net liquidity shock  $L_n$  follows a random walk, namely  $\rho = 0$ . Also, set  $h = 1$  day  $\sigma = 4$ . (This volatility equals 4% of the investor population.)

Figure 2 shows the optimal position size taken by arbitrageurs as a function of the mispricing when the bonds mature in two years. For relatively small levels of mispricing, arbitrageurs do not take any position: the potential profits are not large enough relative to the potential accumulation of holding costs. For larger levels of mispricing, optimal position size increases with the mispricing. The reader can consult Tuckman and Vila (1992a) for further discussion on the properties of optimal arbitrage positions.

INSERT FIGURE 2

Figure 3 shows the equilibrium mispricing with and without arbitrageurs as a function of the net liquidity shock. Again, the bonds have two years left to maturity.

INSERT FIGURE 3

For relatively small net liquidity shocks, in absolute value, the difference between the red and green bond prices increases with the absolute value of the net liquidity shock. Furthermore, the mispricing is the same whether arbitrageurs exist or not. Arbitrageurs make no difference here because, as seen from figure 2, they do not take positions when the mispricing is relatively small.

For larger net liquidity shocks, in absolute value, the mispricing continues to increase with the absolute value of the net liquidity shock, but arbitrage activity makes a difference. Arbitrage activity lowers the mispricing substantially below that which would exist were there no arbitrageurs. Note that the mispricing with arbitrageurs flattens out at the maximum mispricing consistent with no riskless arbitrage opportunities. This bound equals the present value of the holding cost incurred by maintaining an arbitrage position until maturity.

Two lessons emerge from figure 3. First, when the equilibrium mispricing without arbitrageurs is not zero, the equilibrium mispricing with arbitrageurs is also not zero. Due to holding costs and risk aversion, arbitrageurs bring prices closer into line but never eliminate mispricing altogether. Second, arbitrageurs often reduce mispricing to a level below that which would trigger riskless arbitrage transactions. Therefore, market prices will commonly reflect these smaller mispricings and the no-riskless arbitrage bounds will not be binding.

Table 1 summarizes the effects of arbitrage activity on relative mispricings. Conditional on having five years to maturity, one can calculate the expected absolute mispricing and the standard deviation of the mispricing at the two year maturity mark. With two years left to maturity, the maximum absolute price deviation consistent with no-riskless arbitrage is .0089 per dollar face value. As reported in table 1, however, the average absolute price deviation in the presence of arbitrageurs is only .0055. In fact, the mean absolute price deviation without arbitrageurs is also below the no-riskless arbitrage bound. In short, the no-riskless arbitrage condition may not be very useful in describing prices in markets with frictions.

INSERT TABLE 1

#### 4.2. Dynamics of the mispricing process

This subsection studies the effects of arbitrageurs on the dynamics of the mispricing process.

Define the process  $Z_n$  as

$$Z_n = L_n - 2x_n. \quad (17)$$

Note that  $Z_n$  represents the liquidity imbalance between the two markets in the presence of arbitrageurs. Since  $\Delta_n = \theta_n L_n$  without arbitrageurs and  $\Delta_n = \theta_n Z_n$  with arbitrageurs, comparing the processes  $L_n$  and  $Z_n$  is equivalent to comparing the mispricing process with and without arbitrageurs. This comparison is easier than studying the mispricing directly because  $L_n$  has very simple conditional moments:

$$E(dL_n | L_n) = -\rho L_n dt \quad \text{and} \quad \text{Var}(dL_n | L_n) = \sigma^2 dt.$$

These moments reveal that the effect of arbitrage activity can be seen by comparing

$$E \left[ -\frac{dZ_n}{Z_n dt} \mid L_n \right] \quad \text{and} \quad \sqrt{\frac{\text{Var} \left[ \frac{dZ_n}{Z_n dt} \mid L_n \right]}{dt}}$$

with  $\rho$  and  $\sigma$  respectively.

First set the parameter values  $h = 1$  day,  $\rho = .5$  and  $\sigma = 4$ . For the five year bond with a remaining maturity of two years, figure 4 graphs the mean reversion as a function of  $L_n$ , i.e

$$E \left[ -\frac{dZ_n}{Z_n dt} \mid L_n \right].$$

INSERT FIGURE 4

For large absolute values of  $L_n$ , the equilibrium mispricing  $\Delta_n$  approaches the riskless arbitrage bounds, as shown earlier in figure 3. Similarly,  $Z_n$  approaches some bound for large absolute values of  $L_n$ . Consequently,  $Z_n$  is almost constant in these regions and mean reversion is very small.

At the other extreme, for small absolute values of  $L_n$ , arbitrageurs do not take any positions, so  $Z_n = L_n$  and mean reversion equals  $\rho = .5$ .

As  $L_n$  approaches the value at which arbitrageurs will take a position, there is a discontinuity in the second derivative of the size of the arbitrage position. This kink causes mean reversion to be very large.

To summarize the insights provided by figure 4, when mispricings are very small, the presence of arbitrageurs has no effect on the rate at which mispricings vanish. For larger, but relatively small mispricings, arbitrage activity increases the rate at which mispricings vanish. Finally, for relatively large mispricings, arbitrage activity reduces the rate at which mispricings vanish.

Figure 5 plots the function

$$\sqrt{\frac{\text{Var} [dZ_n | L_n]}{dt}}.$$

INSERT FIGURE 5

Notice that arbitrage reduces the conditional variance of the mispricing process. Since  $\Delta_n$  approaches the riskless arbitrage bounds when  $L_n$  is large in absolute value, the conditional variance of the mispricing and of  $Z_n$  must be particularly low. In other words, the reduction of conditional variance due to the activity of arbitrageurs is most pronounced at high levels of mispricings.

The discussion now turns to the average values of the conditional moments for different values of  $\rho$  and  $\sigma$ . Mathematically, define  $\rho'$  and  $\sigma'$  as

$$\rho' = E_N \left[ E_n \left[ - \frac{dZ_n}{Z_n dt} \mid L_n \right] \mid L_N = 0 \right] \quad (18a)$$

$$\sigma' = E_N \left[ \sqrt{\frac{\text{Var}_n [dZ_n | L_n]}{dt}} \mid L_N = 0 \right]. \quad (18b)$$

Table 2, setting  $h = 1$  day and  $\sigma = 4$ , shows  $\rho'$  and  $\sigma'$  for different values of  $\rho$ .

INSERT TABLE 2

As can be seen from this table, arbitrageurs cause the liquidity imbalance process  $Z_n$  to be less

conditionally volatile and more mean reverting than the cumulative liquidity shock process  $L_n$ . While  $\sigma'$  does not change much with  $\rho$ , it can be seen that  $\rho'$  can be very sensitive to increasing values of  $\rho$ . Hence if liquidity shocks are transitory, arbitrage will eliminate them quickly.

Table 3, setting  $h = 1$  day and  $\rho = \frac{1}{2}$ , presents  $\rho'$  and  $\sigma'$  for different values of  $\sigma$ .

### INSERT TABLE 3

Table 3 reinforces the results in table 2, namely, that arbitrage increases average mean reversion and reduces average conditional volatility. Table 3 adds the insight that this effect will be economically significant only when liquidity shocks are sufficiently volatile.

Tables 2 and 3 show that arbitrageurs facing holding costs are particularly effective if liquidity shocks are *conditionally volatile and transient*. Arbitrageurs like conditionally volatile markets since this volatility creates potential arbitrage profits. Also, arbitrageurs like mean reversion because of the nature of holding costs: transient mispricings allow for profit realization before holding costs have a chance to accumulate.

## V. Conclusion

Many financial models assume that markets are frictionless and that riskless arbitrage opportunities do not exist. They then derive precise, relative pricing implications. In the presence of frictions, however, the assumption that riskless arbitrage opportunities do not exist only bounds market mispricings. This paper suggests a research agenda which seeks to find the equilibrium mispricing in a market with costly arbitrage.

The model of equilibrium arbitrage developed here assumes that noise trading in two segmented markets causes price deviations from fundamental values. Risk-averse arbitrageurs facing holding costs reduce these deviations, but do not eliminate them completely. Furthermore, because arbitrageurs will take risky positions in the hopes of capitalizing on market mispricings, riskless

arbitrage arguments may not provide tight bounds around observed market prices.

The equilibrium prices generated by the model provide a number of insights into the way that arbitrage activity absorbs market imbalances and eliminates deviations from fundamental values. In particular, arbitrage activity reduces mispricings and is most effective in doing so when the underlying disturbances are transient and conditionally volatile.

While this paper has taken a step in the direction of modeling equilibrium arbitrage activity, much work remains. Theoretically, the model would be more satisfying if the investors and noise traders were more rational in their market actions. Empirically, the model or its descendants might be applied to particular markets in order to gain insights into the magnitude of market mispricings and into the process from which these magnitudes emerge.



## References

- Aiyagari, S., and M. Gertler, (1991), "Asset Returns with Transactions Costs and Uninsured Individual Risk: A Stage III Exercise," *Journal of Monetary Economics*, 27, 309-331.
- Boyle, P., and T. Vorst, (1992), "Option Replication in Discrete Time with Transaction Costs," *Journal of Finance*, 47(1), 271-293.
- Brennan, M., and E. Schwartz, (1990), "Arbitrage in Stock Index Futures," *Journal of Business*, vol. 63(1), pt. 2, s7-s31.
- Constantinides, G., (1986), "Capital Market Equilibrium with Transaction Costs," *Journal of Political Economy*, 94(4), december, 842-862.
- Davis, M. and A. Norman, (1990), "Portfolio Selection with Transaction Costs," *Mathematics of Operations Research*, .
- Davis, M., V. Panas, and T. Zariphopoulou, (1991), "European Option Pricing with Transaction Costs," mimeo.
- De Long, J., A. Shleifer, L.H. Summers, and R. J. Waldmann, 1990, "Noise trader risk in financial markets," *Journal of Political Economy*, 98, 703-738.
- Duffie, D. and T. Sun (1989) "Transactions Costs and Portfolio Choice in a Discrete-Continuous Time Setting" mimeo, Stanford University.
- Fleming, W., S. Grossman, J.-L. Vila, T. Zariphopoulou, (1989), "Optimal Portfolio Rebalancing with Transaction Costs," unpublished manuscript.
- Fremault, A., (1990), "Execution Lags and Imperfect Arbitrage: The Case of Stock Index Arbitrage," Boston University School of Management Working Paper 9072.
- Fremault, A., (1991), "Stock Index Futures and Index Arbitrage in a Rational Expectations Model," *Journal of Business*, 64 (4), 523-547.
- Grossman, S., and M. Miller, (1988), "Liquidity and Market Structure," *Journal of Finance*, 43, 617-633.
- Grossman, S., and J.-L. Vila, (1992), "Optimal Dynamic Trading with Leverage Constraints," *Journal of Financial and Quantitative Analysis*, (forthcoming).
- He, H. and N. Pearson, (1991a) "Consumption and Portfolio Policies with Incomplete Markets and Short Sales Constraints: The Infinite Dimensional Case," *Journal of Economic Theory*, 54, 259-304.
- He, H. and N. Pearson, (1991b) "Consumption and Portfolio Policies with Incomplete Markets and Short Sales Constraints: The Finite Dimensional Case," *Mathematical Finance*, .

Heaton, J., and D. Lucas, (1992), "Evaluating the Effects of Incomplete Markets on Risk Sharing and Asset Pricing" mimeo, Massachusetts Institute of Technology.

Hodges, S., and A. Neuberger, (1989), "Optimal Replication of Contingent Claims Under Transactions Costs," *The Review of Futures Market*, .

Holden, C., (1990), "A Theory of Arbitrage Trading in Financial Market Equilibrium," Discussion Paper #478, Indiana University.

Lee, C.M., A. Shleifer, and R. H. Thaler, (1991), "Investor Sentiment and the Closed-End Fund Puzzle," *Journal of Finance*, 46, 75-109.

Leland, H., (1985), "Option Pricing and Replication with Transaction Costs," *Journal of Finance*, 44, 1283-1301.

Nelson, D. and K. Ramaswamy (1990) "Simple Binomial Processes as Diffusion Approximations in Financial Models," *Review of Financial Studies*, 3 (3), 393-430.

Shleifer, A, and Vishny, R, (1990), "Equilibrium Short Horizons of Investors and Firms," *American Economic Review Papers and Proceedings*, Vol. 80 #2, 148-153.

Tuckman, B. and J.-L. Vila (1992a), "Arbitrage with Holding Costs: A Utility-Based Approach," *Journal of Finance*, (forthcoming).

Tuckman, B. and J.-L. Vila (1992b), "Grandfather Clauses and Optimal Portfolio Revision," *Journal of Risk and Insurance*, (forthcoming).

Vayanos, D. and J.-L. Vila (1992), "Equilibrium Interest Rate and Liquidity Premium under Proportional Transactions Costs," Massachusetts Institute of Technology, mimeo.

Vila, J.-L. and T. Zariphopoulou, (1990), "Optimal Consumption and Investment with Borrowing Constraints," mimeo, Massachusetts Institute of Technology.

**Table 1**

Table 1: The effects of arbitrage activity on mispricing. Table values are per dollar face value when the bonds have 2 years to maturity. The expected absolute mispricings and the unconditional standard deviation are conditional on having 5 years to maturity.

	Without Arbitrageurs	With Arbitrageurs	% Reduction because of Arbitrageurs
Expected Absolute Mispricing	.0079	.0054	30.4%
Unconditional S.D. of the Mispricing	.010	.0061	37.9%

**Table 2**

Table 2: Average mean reversion and average conditional volatility for different values of  $\rho$ . The conditional volatility of the cumulative liquidity shock,  $L_n$  is set equal to 4.

	$\rho'$	$\sigma'$
$\rho = 0$	.10	2.36
$\rho = \frac{1}{8}$	.23	2.58
$\rho = \frac{1}{2}$	.60	2.85
$\rho = 2$	33	2.44

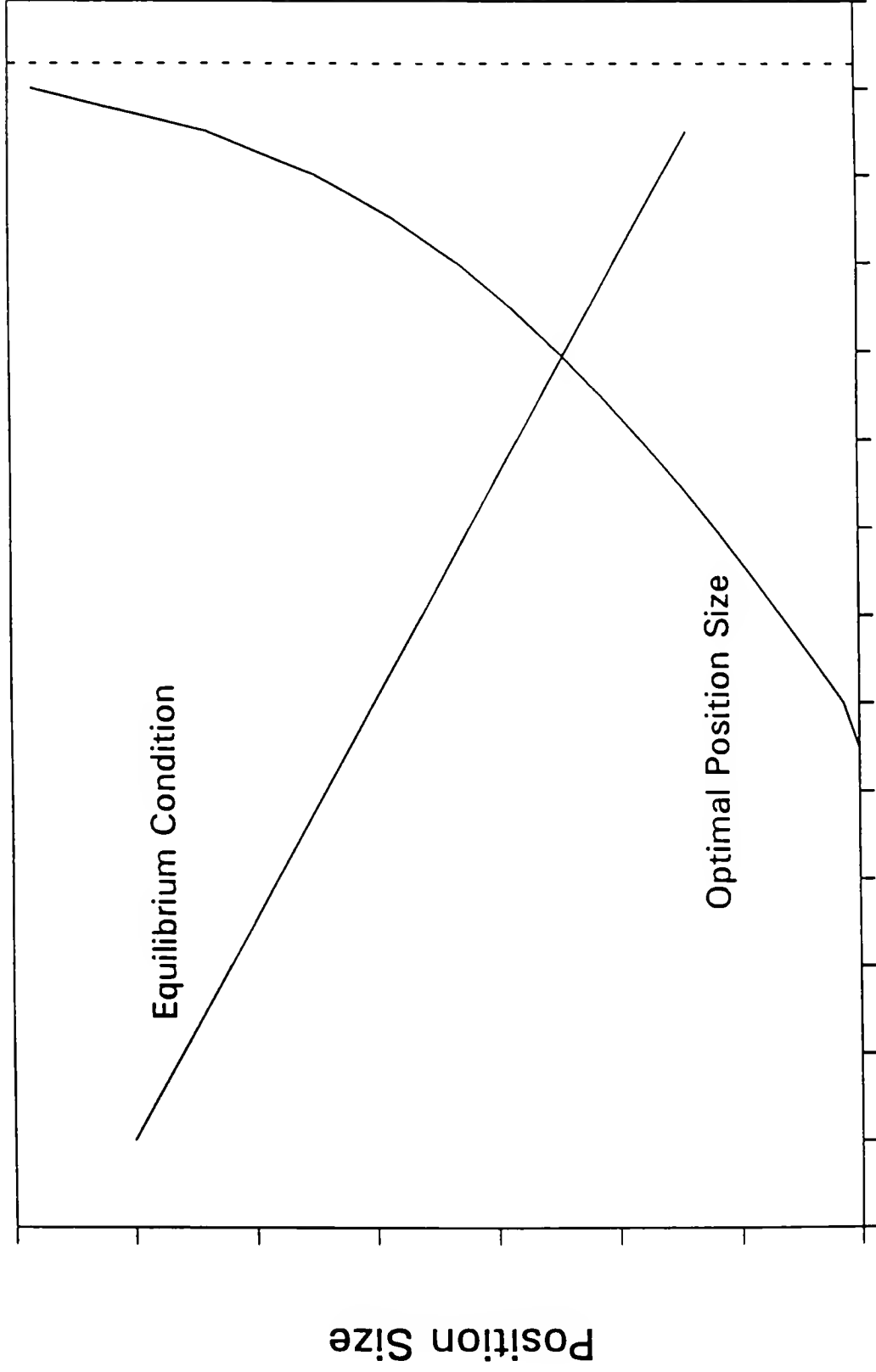
**Table 3**

Table 3: Average mean reversion and conditional volatility for different values of  $\sigma$ . The coefficient of mean reversion of the cumulative liquidity shock,  $\rho$ , is set equal to .5.

	$\rho'$	$\sigma'$	$(\sigma - \sigma')/\sigma$
$\sigma = 1$	.50	1.00	0%
$\sigma = 2$	.50	1.91	.4%
$\sigma = 4$	.60	2.85	28%
$\sigma = 6$	15	3.12	47%

**Figure 1**

# Determination of Equilibrium

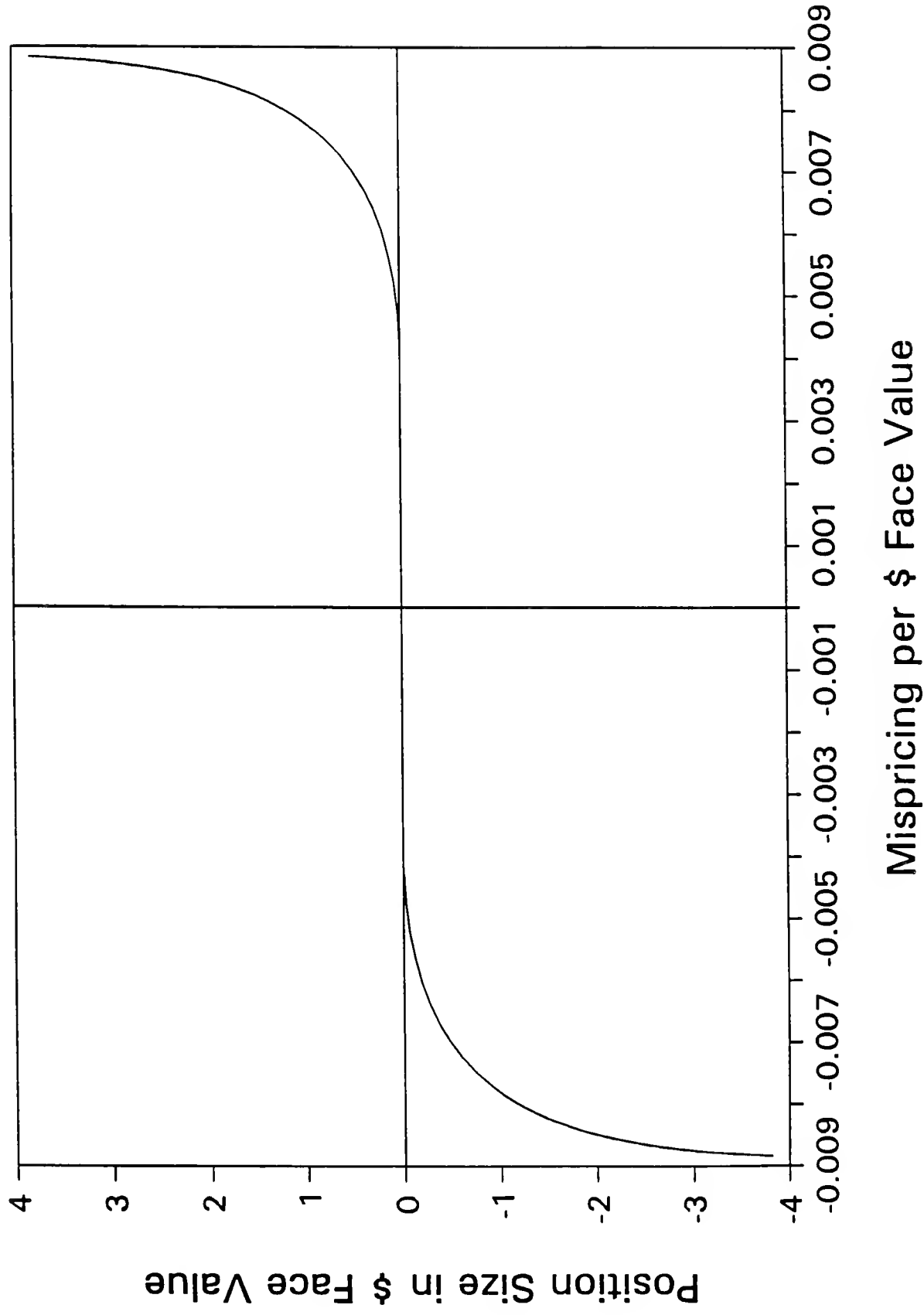


Mispricing

**Figure 2**

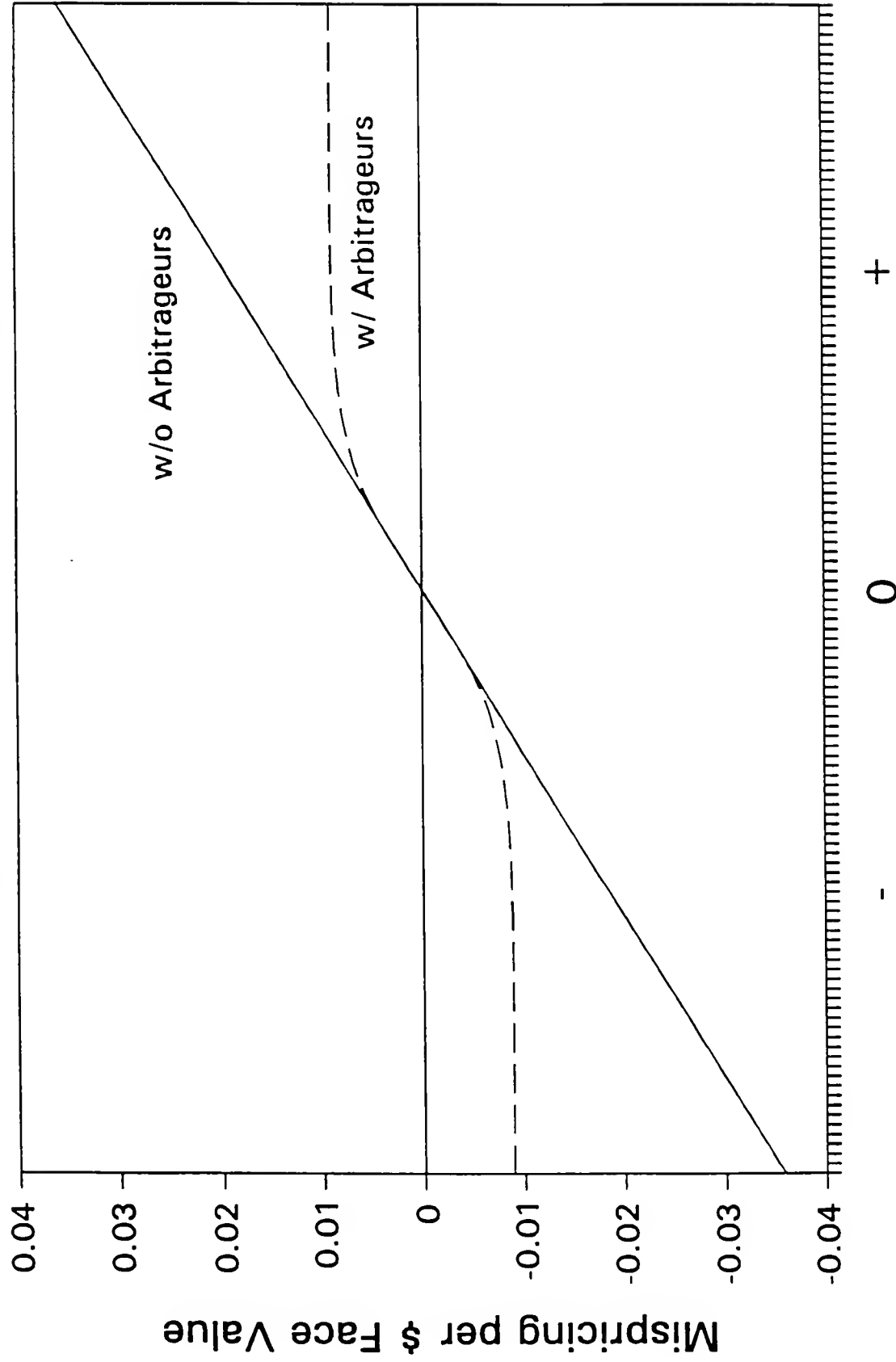


# Equilibrium Position Size



**Figure 3**

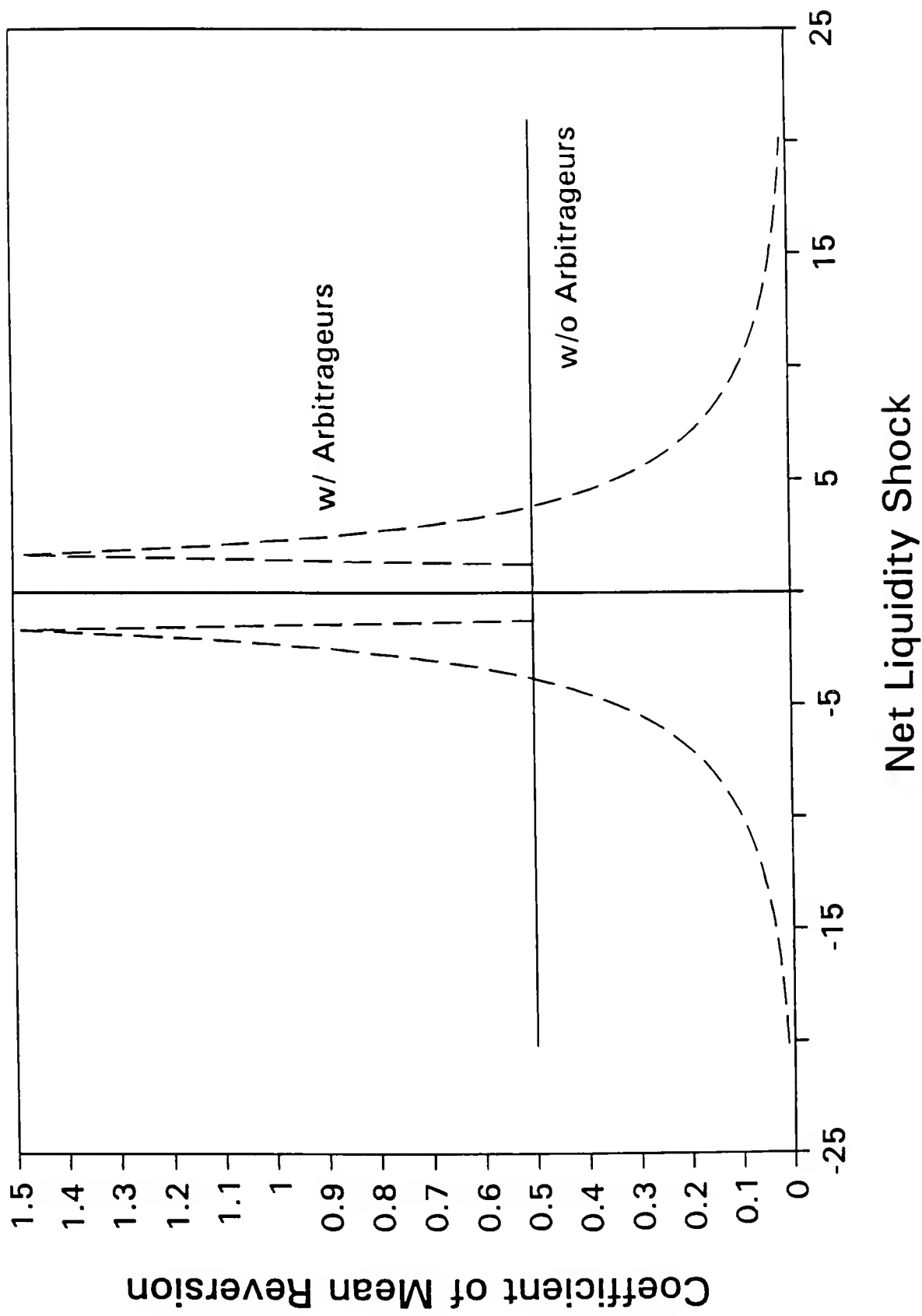
# Mispricing w/ and w/o Arbitrageurs



Net Liquidity Shock

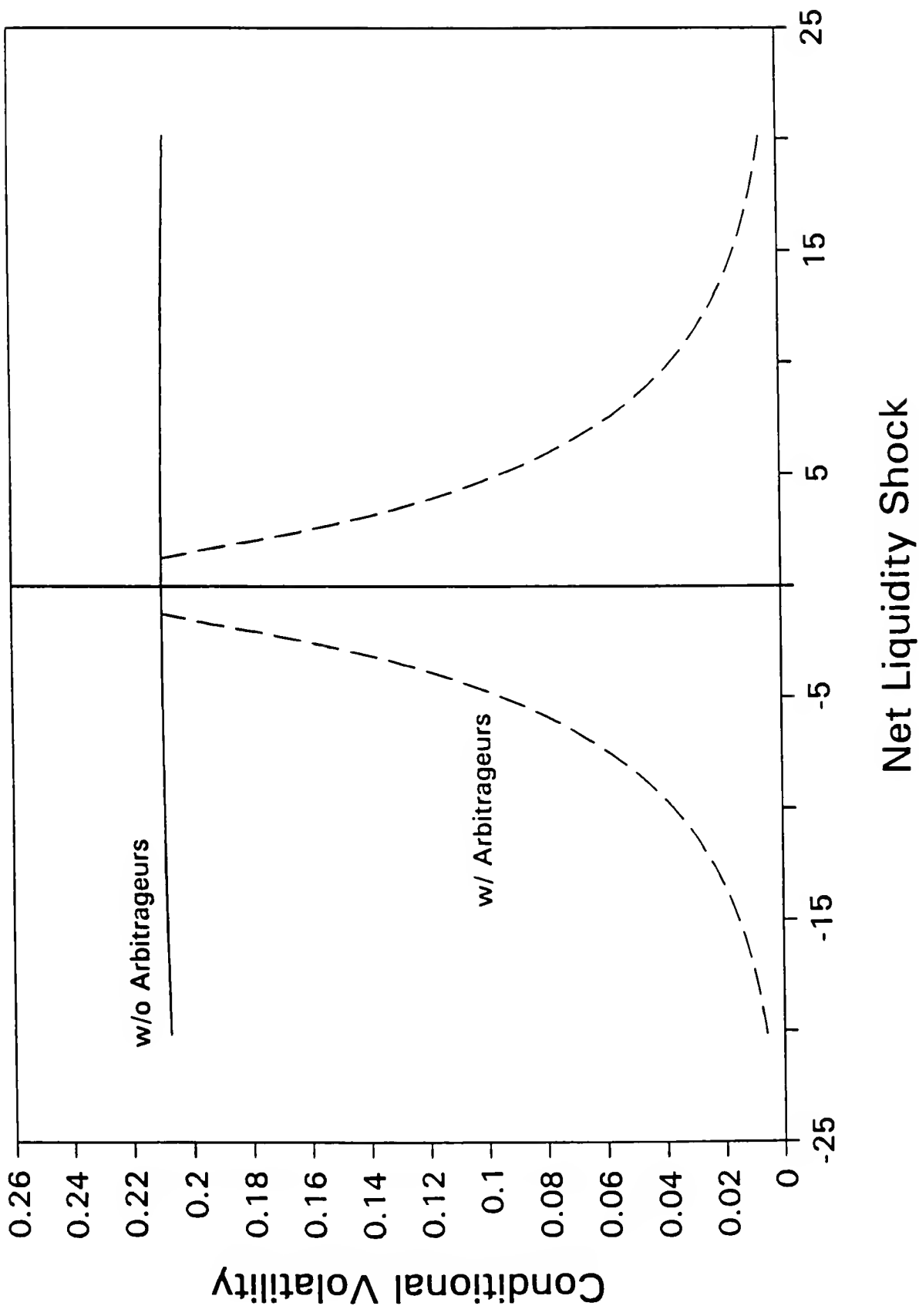
**Figure 4**

# Mean Reversion w/ and w/o Arbitrageurs



**Figure 5**

# Conditional Volatility w/ and w/o Arbitrageurs













# Date Due

MAY 01 1993

Lib-26-67

MIT LIBRARIES DUPL



3 9080 00720872 8

